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Ultradistributions, II
The kernel theorem and ultradistributions with support in a submanifold

By

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The purpose of this paper is to prove the analogues for ultradistributions of two major theorems of L. Schwartz for distributions: the kernel theorem [12], [13], [14], [15] and the structure theorem of distributions with support in a submanifold [11]. We obtain also a Whitney type extension theorem for ultradifferentiable functions as the dual of the second theorem.

This is the second part of our study of ultradistributions and we use the same notations as the first part [5], which we quote as [I]. However, we change the terminology of locally convex spaces a little. According to [6] we call a nuclear space a Grothendieck space and an S^*-space a Komura space. Therefore, an (FN)-space in [I] is called an (FG)-space (= Fréchet-Grothendieck space). An (LFG)-space is the strict inductive limit of a sequence of (FG)-spaces and a (DLFG)-space is the strong dual of an (LFG)-space.

M_p, $p = 0, 1, 2, \ldots$, is a sequence of positive numbers satisfying the following conditions:

(M.0)

(0.1) $$M_0 = 1 ;$$

(M.1) (Logarithmic convexity)

(0.2) $$M_p^2 \leq M_{p-1}M_{p+1}, \quad p = 1, 2, \ldots ;$$

(M.2) (Stability under ultradifferential operators) There are constants A and H such that

$$M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q}, \quad p = 0, 1, \dots ; \tag{0.3}$$

(M.3) (Strong non-quasi-analyticity) There is a constant A such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq Ap\frac{M_p}{M_{p+1}}, \quad p = 1, 2, \dots . \tag{0.4}$$

(M.2) and (M.3) may sometimes be replaced by the following weaker conditions:

(M.2)' (Stability under differential operators) There are constants A and H such that

$$M_{p+1} \leq AH^p M_p, \quad p = 0, 1, \dots; \tag{0.5}$$

(M.3)' (Non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty . \tag{0.6}$$

An infinitely differentiable function φ on an open set Ω in $\mathbb{R}^n$ is said to be an <u>ultradifferentiable function of class</u> (M_p) (resp. $\{M_p\}$) if for each compact set K in Ω and $h > 0$ there is a constant C (resp. there are constants h and C) such that

$$\sup_{x\in K} |D^\alpha \varphi(x)| \leq Ch^{|\alpha|} M_{|\alpha|}, \quad |\alpha| = 0, 1, 2, \dots . \tag{0.7}$$

The space of all ultradifferentiable functions of class (M_p) (resp. $\{M_p\}$) on Ω is denoted by $\mathcal{E}^{(M_p)}(\Omega)$ (resp. $\mathcal{E}^{\{M_p\}}(\Omega)$). The spaces $\mathcal{E}^{(M_p)}(\Omega)$ and $\mathcal{E}^{\{M_p\}}(\Omega)$ as well as

$$\mathcal{D}^{(M_p)}(\Omega) = \mathcal{D}(\Omega) \cap \mathcal{E}^{(M_p)}(\Omega) \quad \text{and} \quad \mathcal{D}^{\{M_p\}}(\Omega) = \mathcal{D}(\Omega) \cap \mathcal{E}^{\{M_p\}}(\Omega) \tag{0.8}$$

have natural locally convex topologies.

An element of the dual $\mathcal{D}^{(M_p)\prime}(\Omega)$ (resp. $\mathcal{D}^{\{M_p\}\prime}(\Omega)$) of $\mathcal{D}^{(M_p)}(\Omega)$ (resp. $\mathcal{D}^{\{M_p\}}(\Omega)$) is called an <u>ultradistibution of class</u> (M_p) (resp. $\{M_p\}$).

The <u>associated function</u>

(0.9) $$M(\rho) = \sup_p \log(\rho^p/M_p), \quad 0 \leqq \rho < \infty$$

plays a very important rôle in our theory. If ζ is a vector in $\mathbb{C}^n$, we write

(0.10) $$M(\zeta) = M(|\zeta|).$$

In § 1 we prove the Paley-Wiener theorem for ultradistributions saying that an entire function $\tilde{f}(\zeta)$ on $\mathbb{C}^n$ is the Fourier-Laplace transform of an ultradistribution f with support in a compact convex set K in $\mathbb{R}^n$ if and only if it satisfies the estimate

(0.11) $$|\tilde{f}(\zeta)| \leq C \exp\{M(L\zeta) + H_K(\zeta)\}, \quad \zeta \in \mathbb{C}^n,$$

where $H_K(\zeta)$ is the <u>support function</u> of K defined by

(0.12) $$H_K(\zeta) = \sup_{x \in K} \operatorname{Im} \langle x, \zeta \rangle.$$

C. Roumieu [10] and M. Neymark [7] have obtained a similar theorem with the right hand side of (0.11) replaced by $C_\varepsilon \exp\{M(L\zeta) + H_K(\zeta) + \varepsilon|\zeta|\}$ for any $\varepsilon > 0$. We eliminate the term $\varepsilon|\zeta|$ with the help of the Phragmén-Lindelöf theorem. In this process condition (M.3) plays an essential role. As Roumieu shows in [9] we cannot obtain estimate (0.11) in general without conditions (M.2) and (M.3).

Section 2 is devoted to the proof of the kernel theorem. Our proof is similar to that of F. Treves [16] in the case of distributions. Condition (M.2) is important in this section.

We write a point in $\mathbb{R}^n = \mathbb{R}^{n'} \times \mathbb{R}^{n''}$ as (x, y) with $x \in \mathbb{R}^{n'}$ and $y \in \mathbb{R}^{n''}$. Let

(0.13) $$F = \{(x, 0);\ x \in \mathbb{R}^{n'},\ 0 \in \mathbb{R}^{n''}\}$$

be a linear submanifold in $\mathbb{R}^n$. We prove in § 3 that $f(x, y)$ is an ultradistribution with support in F if and only if it is developed in the convergent series

$$f(x, y) = \sum_{\beta} f_{\beta}(x) \otimes D^{\beta} \delta(y). \tag{0.14}$$

Roumieu [10] proves that if $f(x, y)$ is an ultradistribution of class $\{M_p\}$ with support in F, then it has the development (0.14) which converges in the topology of ultradistributions of class $\{\sqrt{p!M_p}\}$. We prove the convergence in the topology of the original class.

In the last § 4 we prove that if an infinitely differentiable function φ in the sense of Whitney on a smooth submanifold F satisfies an estimate of class (M_p) (resp. $\{M_p\}$) then it can be extended to an ultradifferentiable function of the same class on a neighborhood of F. This generalizes L. Carleson's theorem [1] in the one-dimensional case. We prove this by showing that the theorem of § 3 is equivalent to this theorem together with the fact that every ultradifferentiable function whose derivatives all vanish on F can be approximated by ultradifferentiable functions whose support does not meet F. We note that the last fact is by no means trivial.

We will employ the theorem of § 3 to characterize those weakly hyperbolic operators for which the Cauchy problem is correctly posed in a Gevrey class of ultradifferentiable functions and ultradistributions.

1. The Paley-Wiener theorem for ultradistributions. Suppose that f is an ultradistribution with compact support in $\mathbb{R}^n$. For each $\zeta \in \mathbb{C}^n$ the function $\exp(-i\zeta x)$ in x belongs to $\mathcal{E}^*(\mathbb{R}^n)$ and it is easily shown that $\exp(-i\zeta x)$ depends on ζ holomorphically in the topology of $\mathcal{E}^*(\mathbb{R}^n)$. Hence

$$\tilde{f}(\zeta) = \langle \exp(-i\zeta x), f(x) \rangle \tag{1.1}$$

defines an entire function on $\mathbb{C}^n$, which we call the Fourier-Laplace transform of f.

The Paley-Wiener theorem holds also for ultradistributions. The associated function $M(\zeta)$ and the support function $H_K(\zeta)$ have the same meaning as in §3 of [I]. The asterisk $*$ stands for either (M_p) or $\{M_p\}$.

Theorem 1.1. Suppose that M_p satisfies conditions (M.0), (M.1), (M.2)' and (M.3)' and that K is a compact convex set in $\mathbb{R}^n$. Then the following conditions are equivalent for an entire function $\tilde{f}(\zeta)$ on $\mathbb{C}^n$:

(a) $\tilde{f}(\zeta)$ is the Fourier-Laplace transform of an ultradistribution $f \in \mathcal{D}^{(M_p)'}_K$ (resp. $\mathcal{D}^{\{M_p\}'}_K$) with support in K;

(b) There are constants L and C (resp. for each $L > 0$ there is a constant C) such that

$$|\tilde{f}(\xi)| \leq C \exp M(L\xi), \quad \xi \in \mathbb{R}^n, \tag{1.2}$$

and for each $\varepsilon > 0$ there is a constant C_ε such that

$$|\tilde{f}(\zeta)| \leq C_\varepsilon \exp\{H_K(\zeta) + \varepsilon|\zeta|\}, \quad \zeta \in \mathbb{C}^n. \tag{1.3}$$

If M_p satisfies (M.2) and (M.3) in addition, then they are also equivalent to the following:

(c) There are constants L' and C (resp. for each $L' > 0$ there is a constant C) such that

(1.4) $\qquad |\tilde{f}(\zeta)| \leq C \exp\{M(L'\zeta) + H_K(\zeta)\}, \quad \zeta \in \mathbb{C}^n.$

A subset B of $\mathcal{D}^{*\prime}_K$ is bounded in $\mathcal{D}^{*\prime}(\mathbb{R}^n)$ if and only if we can choose constants L and C (resp. for each $L > 0$ a constant C) independent of $f \in B$ such that (1.2) holds.

A sequence $f_j \in \mathcal{D}^{(M_p)\prime}_K$ (resp. $\mathcal{D}^{\{M_p\}\prime}_K$) converges if and only if for some L (resp. for any $L > 0$)

(i) $\exp(-M(L\xi))\tilde{f}_j(\xi)$ converges uniformly on $\mathbb{R}^n$.

If M_p satisfies (M.2) and (M.3), then this is also equivalent to each one of the following:

(ii) $\exp(-M(L\zeta))\tilde{f}_j(\zeta)$ converges uniformly on a strip $|\operatorname{Im}\zeta| \leq a < \infty$;

(iii) $\exp\{-M(L\zeta) - H_K(\zeta)\}\tilde{f}_j(\zeta)$ converges uniformly on $\mathbb{C}^n$.

Proof. (a) $\Longrightarrow$ (b). Suppose that B is a bounded set in $\mathcal{D}^{*\prime}(\mathbb{R}^n)$ included in $\mathcal{D}^{*\prime}_K$. By Proposition 5.11 of [I], it is also bounded in the dual $\mathcal{E}^{*\prime}(\mathbb{R}^n)$ of the reflexive space $\mathcal{E}^*(\mathbb{R}^n)$. Hence there are a regular compact set K_1 in $\mathbb{R}^n$ and constants h and C (resp. and for each $h > 0$ a constant C) independent of $f \in B$ such that

$$(1.5) \qquad |\langle \varphi, f\rangle| \leq C \sup_{\substack{x\in K_1\\ \alpha}} \frac{|D^\alpha \varphi(x)|}{h^{|\alpha|}M_{|\alpha|}}, \qquad \varphi \in \mathcal{E}^*(\mathbb{R}^n).$$

If we take $\varphi(x) = \exp(-ix\xi)$, $\xi \in \mathbb{R}^n$, then the right hand side of (1.5) is bounded by

$$C \sup \frac{|\xi^\alpha|}{h^{|\alpha|}M_{|\alpha|}} \lesssim C \sup \frac{|\xi|^{|\alpha|}}{h^{|\alpha|}M_{|\alpha|}} = C \exp M(\xi/h).$$

Hence we have (1.2).

Since ultradistributions are imbedded in the hyperfunctions without changing the support, (1.3) follows from the Ehrenpreis-Martineau theorem (Hörmander [4], Theorem 4.5.3). We will give here a direct proof, however.

Let $\varepsilon > 0$ and let K_ε be the set of all points x in $\mathbb{R}^n$ such that the distance from x to K is less than or equal to ε. For each $h > 0$ we can find an ultradifferentiable function $\chi(x) \in \mathcal{D}^{\{M_p\},h/2}(\mathbb{R}^n)$ with support in the interior of K_ε which takes the value 1 on a neighborhood of K. Then we have

$$\langle \varphi, f \rangle = \langle \chi\varphi, f \rangle, \quad \varphi \in \mathcal{E}^*(\mathbb{R}^n), \quad f \in B.$$

In view of [I], Proposition 2.7 we have for some C_1

$$|\langle \varphi, f \rangle| \leq C \sup_{\substack{x \in K_1 \\ \alpha}} \frac{|D^\alpha(\chi\varphi)(x)|}{h^{|\alpha|} M_{|\alpha|}}$$

$$\leq C_1 \sup_{\substack{x \in K_\varepsilon \\ \alpha}} \frac{|D^\alpha \varphi(x)|}{(h/2)^{|\alpha|} M_{|\alpha|}}, \quad \varphi \in \mathcal{E}^*(\mathbb{R}^n).$$

Let $\varphi(x) = \exp(-i\zeta x)$ with $\zeta \in \mathbb{C}^n$. Then we obtain

$$|\tilde{f}(\zeta)| \leq C_1 \exp\{M(2\zeta/h) + H_{K_\varepsilon}(\zeta)\}.$$

Since $H_{K_\varepsilon}(\zeta) \leq H_K(\zeta) + \varepsilon|\zeta|$ and since $M(\rho) = o(\rho)$ as $\rho \to \infty$ ([I], (4.7)), this implies (1.3).

(b) $\Longrightarrow$ (a). Suppose that $\tilde{B}$ is a set of entire functions $\tilde{f}(\zeta)$ on $\mathbb{C}^n$ satisfying (1.2) and (1.3) with a uniform constant C.

If $\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R}^n)$ (resp. $\mathcal{D}^{\{M_p\}}(\mathbb{R}^n)$), then it follows from the Paley-Wiener theorem for ultradifferentiable functions ([I], Theorem 9.1) that for each $h > 0$ there is a constant C (resp. there are constants h and C) such that the Fourier-Laplace transform $\tilde{\varphi}(\zeta)$ satisfies

(1.6) $$|\tilde{\varphi}(\zeta)| \leq C_1 \exp\{-M(\zeta/h) + H_{K_1}(\zeta)\},$$

where K_1 is the convex hull of supp φ.

Hence for each $\tilde{f} \in \tilde{B}$

$$|\tilde{\varphi}(\xi)\tilde{f}(-\xi)| \leq C_1 C \exp\{M(L\xi) - M(\xi/h)\}$$

is integrable on $\mathbb{R}^n$ by Proposition 3.4 of [I]. Thus

$$\langle \varphi, f \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{\varphi}(\xi)\tilde{f}(-\xi)d\xi \tag{1.7}$$

defines a linear functional f on $\mathcal{D}^*(\mathbb{R}^n)$. The boundedness of $\{f;\ \tilde{f} \in \tilde{B}\}$ follows from the proof because we can choose a uniform constant C_1 in (1.7) for all φ in a bounded set in $\mathcal{D}^*(\mathbb{R}^n)$.

The fact that $\operatorname{supp} f \subset K$ may also be proved by the Ehrenpreis-Martineau theorem but we prove it directly.

Suppose that $\varphi \in \mathcal{D}^*_{K_1}$, where K_1 is a compact convex set in $\mathbb{R}^n$ with $K \cap K_1 = \phi$. We choose a $\delta > 0$ smaller than the distance between K_1 and K. Then there exists a real unit vector ξ_0 such that

$$H_{K_1}(i\xi_0) + H_K(-i\xi_0) = \sup_{x\in K_1}\langle x, \xi_0\rangle - \inf_{x\in K}\langle x, \xi_0\rangle < -\delta .$$

We consider for each $\xi \in \mathbb{R}^n$ the holomorphic function

$$E(z) = \tilde{\varphi}(\xi + \xi_0 z)\tilde{f}(-\xi - \xi_0 z)$$

defined on the upper half plane $\operatorname{Im} z \geq 0$. By (1.2), (1.3) and (1.6) we have

$$|F(x)| \leq C_1 C \exp\{-M((\xi + \xi_0 x)/h) + M(L(\xi + \xi_0 x))\}, \quad x \in \mathbb{R},$$

and

$$|F(z)| \leq C_1 C_\varepsilon \exp\{-M((\xi + \xi_0 z)/h) + H_{K_1}(\xi_0 z) + H_K(-\xi_0 z) + \varepsilon|\xi + \xi_0 z|\}$$

$$\leq C_1 C_\varepsilon \exp\{-M((\xi + \xi_0 z)/h) - \delta \operatorname{Im} z + \varepsilon|\xi| + \varepsilon|z|\}, \quad \operatorname{Im} z \geq 0.$$

Let ξ' be the component of ξ orthogonal to ξ_0 and $\xi = x_0\xi_0 + \xi'$. Then we have by Proposition 3.4 of [I]

$$\exp\{-M((\xi+\xi_0 x)/h)+M(L(\xi+\xi_0 x))\}\leq A(1+|\xi+\xi_0 x|)^{-n-1}$$

$$\leq A\left|(x+x_0+(1+|\xi'|)i)^{-n-1}\right|,\quad x\in\mathbb{R}.$$

Applying the Phragmén-Lindelöf theorem to $(z+x_0+(1+|\xi'|)i)^{n+1}e^{-i\delta z}F(z)$, we have finally

$$|\tilde{\varphi}(\xi+i\tau\xi_0)\tilde{f}(-\xi-i\tau\xi_0)|\leq A_1 e^{-\delta\tau}(1+|\xi|)^{-n-1},\quad \xi\in\mathbb{R}^n,\ \tau\geq 0\ .$$

Thus we can deform the domain of integral of (1.7) to $\mathbb{R}^n+i\tau\xi_0$ and obtain $\langle\varphi, f\rangle = 0$ as $\tau\to\infty$.

Since every $\varphi\in\mathcal{D}^*(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi\cap K=\phi$ can be represented as the sum of a finite number of $\varphi_i\in\mathcal{D}^*(\mathbb{R}^n)$ with the above property, we have $\operatorname{supp} f\subset K$.

In order to prove that $\tilde{f}(\zeta)$ is the Fourier-Laplace transform of f, we consider the regularization

$$(\psi*f)(x_0)=\langle\psi(x_0-x), f(x)\rangle\ ,$$

where $\psi\in\mathcal{D}^*(\mathbb{R}^n)$. Since the Fourier transform of $\psi(x_0-x)$ is equal to $e^{-ix_0\xi}\tilde{\psi}(-\xi)$, we have by (1.7)

$$(\psi*f)(x_0)=\frac{1}{(2\pi)^n}\int e^{ix_0\xi}\tilde{\psi}(\xi)\tilde{f}(\xi)\,d\xi\ .$$

Since $\psi*f$ is a continuous function with compact support ([I], Theorem 6.10), this proves that $\tilde{\psi}(\xi)\tilde{f}(\xi)$ is the Fourier transform of $\psi*f$. Let

$$\psi_\varepsilon(x)=\varepsilon^{-n}\psi_1(x/\varepsilon)$$

with a $\psi_1(x)\in\mathcal{D}^*(\mathbb{R}^n)$ such that $\int\psi_1(x)dx=1$ and that $\psi_1(-x)=\psi_1(x)$. The mapping $f\mapsto\psi_\varepsilon*f$ on $\mathcal{E}^{*\prime}(\mathbb{R}^n)$ into itself is easily shown to be the dual of the mapping $\varphi\mapsto\varphi*\psi_\varepsilon$ on $\mathcal{E}^*(\mathbb{R}^n)$ into itself and the latter converges to the identity mapping as $\varepsilon \to 0$. Hence $\psi_\varepsilon * f$ converges to f in $\mathcal{E}^{*\prime}(\mathbb{R}^n)$ so that $\tilde{\psi}_\varepsilon(\xi)\tilde{f}(\xi)$ converges to the Fourier transform of f. Since $\tilde{\psi}_\varepsilon(\xi)$ converges to one, $\tilde{f}$ must coincide with the Fourier transform of f.

(b) $\Rightarrow$ (c). For each real unit vector ξ_0 we write

$$\delta_0 = H_K(i\xi_0) = \sup_{x \in K} \langle x, \xi_0 \rangle .$$

Then for each $\xi \in \mathbb{R}^n$ the holomorphic function

$$F(z) = \tilde{f}(\xi + \xi_0 z)$$

on the upper half plane $\operatorname{Im} z \geqq 0$ satisfies

$$|F(x)| \leqq C \exp M(L(\xi + \xi_0 x)), \quad x \in \mathbb{R},$$

and

$$|F(z)| \leqq C_\varepsilon \exp\{\delta_0 \operatorname{Im} z + \varepsilon|\xi| + \varepsilon|z|\}, \quad \operatorname{Im} z \geqq 0.$$

Let

$$(1.8) \qquad P(z) = \prod_{p=1}^{\infty} \left(1 - \frac{iL(z + x_0 + |\xi'|i)}{m_p}\right).$$

Then we have by (10.5) of [I]

$$|P(x)^{-1} e^{i\delta_0 x} F(x)| \leqq C, \quad x \in \mathbb{R},$$

and

$$|P(z)^{-1} e^{i\delta_0 z} F(z)| \leqq C_\varepsilon \exp\{\varepsilon|\xi| + \varepsilon|z|\}, \quad \operatorname{Im} z \geqq 0.$$

Hence it follows from the Phragmén-Lindelöf theorem that

$$|F(z)| \leqq C|P(z)e^{-i\delta_0 z}| \leqq C|P(z)|\exp(H_K(\xi + \xi_0 z)).$$

On the other hand, we have by Proposition 4.6 of [I]

$$|P(z)| \leqq A \exp(M(L'(\xi + \xi_0 z)))$$

for some constants A and L'. Consequently (1.4) holds.

Trivially (c) implies (b).

The strong topologies of $\mathcal{D}^{*\prime}(\mathbb{R}^n)$ and $\mathcal{E}^{*\prime}(\mathbb{R}^n)$ coincide on $\mathcal{D}^{*\prime}_K$. In fact, if χ is a function in $\mathcal{D}^*(\mathbb{R}^n)$ which is equal to one on a neighborhood of K, then we have for every $f \in \mathcal{D}^{*\prime}_K$

$$\langle \varphi, f \rangle = \langle \chi\varphi, f \rangle, \quad \varphi \in \mathcal{D}^{*\prime}(\mathbb{R}^n)$$

and the multiplication by χ is continuous on $\mathcal{E}^*(\mathbb{R}^n)$ into $\mathcal{D}^*(\mathbb{R}^n)$.

In particular, $\mathcal{D}^{(M_p)\prime}{}_K$ (resp. $\mathcal{D}^{\{M_p\}\prime}{}_K$) is a (DFS)-space (resp. an (FS)-space as a closed linear subspace of the (DFS)-space $\mathcal{E}^{(M_p)\prime}(\mathbb{R}^n)$ (resp. the (FS)-space $\mathcal{D}^{\{M_p\}\prime}(\mathbb{R}^n)$). In both cases a sequence f_j converges if and only if f_j are contained in an absolutely convex compact set B and f_j converges in norm of the Banach space X_B generated by B ([6], Theorem I.12.3 and Theorem III, 9.5). In case $* = (M_p)$, the least constant C of (1.2) is exactly the norm of X_B for some B. In case $* = \{M_p\}$, the above proof shows that (i) implies the uniform convergence of f_j on every bounded set in $\mathcal{D}^*(\mathbb{R}^n)$. The converse is clear.

If M_p satisfies (M.2) and (M.3), then the proof of the part (b) $\Rightarrow$ (c) shows that (i) implies (iii). The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial.

2. The kernel theorem. We say that a subset K of $\mathbb{R}^n$ satisfies the cone property if for each $x \in K$ there are a neighborhood $U \cap K$ of x, a unit vector e in $\mathbb{R}^n$ and a positive number ε_0 such that $(U \cap K) + \varepsilon e$ is in the interior of K for any $0 < \varepsilon < \varepsilon_0$.

In this section we denote by Ω' and Ω'' open sets in $\mathbb{R}^{n'}$ and $\mathbb{R}^{n''}$ respectively. A point in Ω' (resp. in Ω'') is denoted by x (resp. y). Similarly we denote by K' and K'' compact sets with the cone property included in Ω' and Ω'' respectively.

Theorem 2.1. Suppose that M_p satisfies (M.0), (M.1), (M.2) and (M.3)'. Then the bilinear mapping which assigns to each pair of functions $\varphi(x)$ on Ω' and $\psi(y)$ on Ω'' the product $\varphi(x)\psi(y)$ on $\Omega' \times \Omega''$ induces the following isomorphisms of locally convex spaces:

$$\mathcal{E}^{(M_p)}(\Omega') \hat{\otimes} \mathcal{E}^{(M_p)}(\Omega'') \cong \mathcal{E}^{(M_p)}(\Omega' \times \Omega''); \tag{2.1}$$

$$\mathcal{E}^{\{M_p\}}(\Omega') \hat{\otimes} \mathcal{E}^{\{M_p\}}(\Omega'') \cong \mathcal{E}^{\{M_p\}}(\Omega' \times \Omega''); \tag{2.2}$$

$$\mathcal{D}^{(M_p)}_{K'} \hat{\otimes} \mathcal{D}^{(M_p)}_{K''} \cong \mathcal{D}^{(M_p)}_{K' \times K''}; \tag{2.3}$$

$$\mathcal{D}^{\{M_p\}}_{K'} \hat{\otimes} \mathcal{D}^{\{M_p\}}_{K''} \cong \mathcal{D}^{\{M_p\}}_{K' \times K''}; \tag{2.4}$$

$$\mathcal{D}^{\{M_p\}}(\Omega') \hat{\otimes} \mathcal{D}^{\{M_p\}}(\Omega'') \cong \mathcal{D}^{\{M_p\}}(\Omega' \times \Omega''). \tag{2.5}$$

Proof. Since $\mathcal{E}^*(\Omega)$, $\mathcal{D}^*_K$ and $\mathcal{D}^*(\Omega)$ are Grothendieck spaces ([I], Theorem 2.6), the projective topology π and the biequicontinuous topology ε coincide on the tensor products.

Since the polynomials are dense in $\mathcal{E}^*(\Omega)$ ([I], Theorem 7.3), $\mathcal{E}^*(\Omega') \otimes \mathcal{E}^*(\Omega'')$ is dense in $\mathcal{E}^*(\Omega' \times \Omega'')$.

The continuity of multiplication $\mathcal{E}^*(\Omega') \times \mathcal{E}^*(\Omega'') \longrightarrow \mathcal{E}^*(\Omega' \times \Omega'')$ ([I], Theorem 2.8) implies that the induced injection $\mathcal{E}^*(\Omega') \otimes_\pi \mathcal{E}^*(\Omega'') \longrightarrow \mathcal{E}^*(\Omega' \times \Omega'')$ is continuous.

To prove that it is an open mapping onto the image, we consider arbitrary equicontinuous sets A in $\mathcal{E}^{*\prime}(\Omega')$ and B in $\mathcal{E}^{*\prime}(\Omega'')$. Then there exist regular compact sets L' in Ω' and L'' in Ω'', and constants h, C' and C'' (resp. for each $h > 0$ constants C' and C'') such that

$$p^A(\varphi) = \sup_{f \in A} |\langle \varphi, f \rangle| \leq C' \sup_{\substack{x \in L' \\ \alpha}} \frac{|D^\alpha \varphi(x)|}{h^{|\alpha|} M_{|\alpha|}}, \tag{2.6}$$

$$(2.7)\qquad p^B(\psi) = \sup_{g\in B} |\langle \psi, g\rangle| \leqq C' \sup_{\substack{y\in L'\\ \beta}} \frac{|D^\beta \psi(y)|}{h^{|\beta|}M_{|\beta|}} .$$

Suppose that $\chi \in \mathcal{E}^*(\Omega') \otimes \mathcal{E}^*(\Omega'')$, $f \in A$ and $g \in B$. Then we have by (M.2)

$$(2.8)\qquad \left|\iint \chi(x, y)f(x)g(y)dxdy\right| \leqq C' \sup_{\substack{x\in L'\\ \alpha}} \frac{|D_x^\alpha \int \chi(x,y)g(y)dy|}{h^{|\alpha|}M_{|\alpha|}}$$

$$\leqq C'C'' \sup_{\substack{(x,y)\in L'\times L''\\ \alpha\ \beta}} \frac{|D_x^\alpha D_y^\beta \chi(x,y)|}{h^{|\alpha+\beta|}M_{|\alpha|}M_{|\beta|}}$$

$$\leqq AC'C'' \sup_{\substack{(x,y)\in L'\times L''\\ \gamma}} \frac{|D_{x,y}^\gamma \chi(x,y)|}{(h/H)^{|\gamma|}M_{|\gamma|}} .$$

Thus the semi-norm

$$(2.9)\qquad p^{A\otimes B}(\chi) = \sup_{h\in A\otimes B} |\langle \chi, h\rangle|$$

is bounded by a continuous semi-norm on $\mathcal{E}^*(\Omega' \times \Omega'')$. In other words, the ε-topology on $\mathcal{E}^*(\Omega') \otimes \mathcal{E}^*(\Omega'')$ is weaker than the induced topology from $\mathcal{E}^*(\Omega' \times \Omega'')$.

Since $\mathcal{E}^*(\Omega' \times \Omega'')$ is complete ([I], Theorem 2.6), we obtain the isomorphisms (2.1) and (2.2).

The proof of (2.5) is similar. To prove that $\mathcal{D}^{\{M_p\}}(\Omega') \otimes \mathcal{D}^{\{M_p\}}(\Omega'')$ is dense in $\mathcal{D}^{\{M_p\}}(\Omega' \times \Omega'')$, let $\varphi \in \mathcal{D}^{\{M_p\}}(\Omega' \times \Omega'')$. Let L' (resp. L") be the projection of $\operatorname{supp}\varphi$ into Ω' (resp. Ω''). We choose a $\chi' \in \mathcal{D}^{\{M_p\}}(\Omega')$ (resp. $\chi'' \in \mathcal{D}^{\{M_p\}}(\Omega'')$) which is equal to one on a neighborhood of L' (resp. L"). By Lemma 7.1 of [I], there is a sequence $\varphi_j \in \mathcal{A}(\Omega' \times \Omega'')$ which converges to φ in $\mathcal{E}^{\{M_p\}}(\Omega' \times \Omega'')$. By approximating φ_j by polynomials, we can find a sequence of polynomials ψ_j which converges to φ in $\mathcal{E}^{\{M_p\}}(L)$, where L is a regular compact neighborhood of $\operatorname{supp}\chi' \times \operatorname{supp}\chi''$. Then it is easy to see that $\chi'(x)\chi''(y)\psi_j(x, y) \in \mathcal{D}^{\{M_p\}}(\Omega') \otimes \mathcal{D}^{\{M_p\}}(\Omega'')$ converges to φ in $\mathcal{D}^{\{M_p\}}(\Omega' \times \Omega'')$.

The multiplication is hypocontinuous on $\mathcal{D}^{\{M_p\}}(\Omega') \times \mathcal{D}^{\{M_p\}}(\Omega'')$ into $\mathcal{D}^{\{M_p\}}(\Omega' \times \Omega'')$ ([I], Theorem 2.8) and $\mathcal{D}^{\{M_p\}}(\Omega')$ and $\mathcal{D}^{\{M_p\}}(\Omega'')$ are (DF)-spaces. Hence the multiplication is continuous by Théorème 2 of Grothendieck [2] and therefore $\mathcal{D}^{\{M_p\}}(\Omega') \otimes_\pi \mathcal{D}^{\{M_p\}}(\Omega'') \to \mathcal{D}^{\{M_p\}}(\Omega' \times \Omega'')$ is continuous.

Let A and B be equicontinuous sets in $\mathcal{D}^{\{M_p\}}{}'(\Omega')$ and $\mathcal{D}^{\{M_p\}}{}'(\Omega'')$ respectively. Then for any regular compact sets L' in Ω' and L'' in Ω'' and for any $h > 0$ we can find constants C' and C'' such that (2.6) and (2.7) hold for all $\varphi \in \mathcal{D}^{\{M_p\}}_{L'}$ and $\psi \in \mathcal{D}^{\{M_p\}}_{L''}$.

We have by the same computation as above that the semi-norm $p^{A \otimes B}$ defined by (2.9) is continuous on $\mathcal{D}^{\{M_p\}}_{L' \times L''}$. Since every compact set in $\Omega' \times \Omega''$ is included in a compact set of the form $L' \times L''$, it follows that $p^{A \otimes B}$ is continuous on $\mathcal{D}^{\{M_p\}}(\Omega' \times \Omega'')$.

Lastly we prove (2.3) and (2.4). Since $\mathcal{D}^*_K$ is regarded as a closed linear subspace of $\mathcal{E}^*(\Omega)$, $\mathcal{D}^*_{K'} \hat{\otimes}_\varepsilon \mathcal{D}^*_{K''}$ is identified with a closed linear subspace R of $\mathcal{E}^*(\Omega' \times \Omega'')$. Clearly R is included in $\mathcal{D}^*_{K' \times K''}$. On the other hand, if the support of $\chi \in \mathcal{D}^*_{K' \times K''}$ is included in a compact set in the interior of $K' \times K''$, then χ can be approximated by a sequence of elements in $\mathcal{D}^*_{K'} \otimes \mathcal{D}^*_{K''}$ as in the proof of (2.5), so that χ belongs to R.

Since $K' \times K''$ has the cone property, such χ form a dense linear subspace of $\mathcal{D}^*_{K' \times K''}$. In fact, let φ be an arbitrary element in $\mathcal{D}^*_{K' \times K''}$. There is a partition of unity $1 = \sum \psi_j$ on a neighborhood of $K' \times K''$ which is subordinate to the open covering associated with the cone property. Every $\psi_j \varphi$ may be translated to a function χ_j with a compact support in the interior of $K' \times K''$. Since the translation is continuous in $\mathcal{D}^*(\Omega)$ as was shown in the proof of Theorem 6.10 of [I], $\varphi = \sum \psi_j \varphi$ is the limit of a sequence of functions in $\mathcal{D}^*_{K' \times K''}$ with compact supports in the interior of $K' \times K''$. Thus R coincides with $\mathcal{D}^*_{K' \times K''}$.

As for (2.4) we have the following more precise results.

Proposition 2.2. *Suppose that* M_p *satisfies* (M.0), (M.1), (M.2) *and* (M.3)' *and that* K' *and* K'' *are compact sets with the cone property. Then for every* $h > 0$ (*resp. every* $k > 0$) *we can find a* $k > h$ (*resp. an* $0 < h < k$) *so that we have the following continuous inclusions of Banach spaces:*

$$\mathcal{D}^{\{M_p\},h}_{K'} \hat{\otimes}_\pi \mathcal{D}^{\{M_p\},h}_{K''} \subset \mathcal{D}^{\{M_p\},h}_{K' \times K''}. \tag{2.10}$$

$$\mathcal{D}^{\{M_p\},h}_{K'} \hat{\otimes}_\varepsilon \mathcal{D}^{\{M_p\},h}_{K''} \subset \mathcal{D}^{\{M_p\},k}_{K'} \hat{\otimes}_\pi \mathcal{D}^{\{M_p\},k}_{K''}; \tag{2.11}$$

$$\mathcal{D}^{\{M_p\},h}_{K' \times K''} \subset \mathcal{D}^{\{M_p\},k}_{K'} \hat{\otimes}_\varepsilon \mathcal{D}^{\{M_p\},k}_{K''}. \tag{2.12}$$

Proof. (2.10) follows from the continuity of the multiplication $\mathcal{D}^{\{M_p\},h}_{K'} \times \mathcal{D}^{\{M_p\},h}_{K'} \longrightarrow \mathcal{D}^{\{M_p\},h}_{K' \times K''}$.

If $T : X \longrightarrow X_1$ is a nuclear linear mapping and $S : Y \longrightarrow Y_1$ is a continuous linear mapping, then $T \otimes S : X \otimes_\varepsilon Y \longrightarrow X_1 \otimes_\pi Y_1$ is continuous (cf. Pietsch [8], Satz 7.3.2). Hence (2.11) is proved by the fact that $\mathcal{D}^{\{M_p\},h}_{K'} \longrightarrow \mathcal{D}^{\{M_p\},k}_{K'}$ is nuclear ([I], Proposition 2.4).

(2.8) shows that the norm of $\mathcal{D}^{\{M_p\},k}_{K'} \otimes_\varepsilon \mathcal{D}^{\{M_p\},k}_{K''}$ is bounded by a constant times the norm of $\mathcal{D}^{\{M_p\},k/H}_{K' \times K''}$. We can prove in the same way as above that $\mathcal{D}^{\{M_p\},k}_{K'} \otimes \mathcal{D}^{\{M_p\},k}_{K''}$ is dense in $\mathcal{D}^{\{M_p\},h}_{K' \times K''}$ in the norm of $\mathcal{D}^{\{M_p\},k/H}_{K' \times K''}$ for some $h < k/H$. We can also start with h.

Let X and Y be locally convex spaces. Then we denote by $B^s(X, Y)$ (resp. $B(X, Y)$) the space of all separately continuous (resp. continuous) bilinear functionals on $X \times Y$. Under a mild condition we can introduce in it the topology of bibounded convergence or the topology of uniform convergence on the sets of the form $A \times B$, where A and B are bounded sets in X and Y respectively. The space $B^s(X, Y)$ (resp. $B(X, Y)$) equipped with this topology is denoted by $B^s_\beta(X, Y)$ (resp. $B_\beta(X, Y)$).

$L(X, Y)$ denotes the space of all continuous linear mappings $T : X \longrightarrow Y$ and $L_\beta(X, Y)$ stands for $L(X, Y)$ equipped with the topology of uniform convergence on the bounded sets in X.

The following is the <u>kernel theorem</u> for ultradistributions.

<u>Theorem 2.3.</u> <u>Suppose that</u> M_p <u>satisfies</u> (M.0), (M.1), (M.2) <u>and</u> (M.3)'. <u>Let</u> $*$ <u>be either</u> (M_p) <u>or</u> $\{M_p\}$. <u>Then we have the canonical isomorphisms of locally convex spaces:</u>

$$\begin{aligned}(2.13)\qquad B^s_\beta(\mathcal{D}^*(\Omega'), \mathcal{D}^*(\Omega'')) &= L_\beta(\mathcal{D}^*(\Omega'), \mathcal{D}^{*\prime}(\Omega''))\\ &= L_\beta(\mathcal{D}^*(\Omega''), \mathcal{D}^{*\prime}(\Omega')) = \mathcal{D}^{*\prime}(\Omega')\,\hat{\otimes}\,\mathcal{D}^{*\prime}(\Omega'') = \mathcal{D}^{*\prime}(\Omega' \times \Omega'').\end{aligned}$$

<u>Proof.</u> Since $\mathcal{D}^*(\Omega')$ and $\mathcal{D}^*(\Omega'')$ are reflexive spaces ([I], Theorem 2.6), $B^s_\beta(\mathcal{D}^*(\Omega'), \mathcal{D}^*(\Omega''))$ is identified with the space $B^s_\varepsilon((\mathcal{D}^{*\prime}(\Omega'))'_{\sigma^*}, (\mathcal{D}^{*\prime}(\Omega''))'_{\sigma^*})$ of separately weak*-continuous bilinear functionals equipped with the topology of bi-equicontinuous convergence. The latter space is canonically isomorphic to $L_\varepsilon((\mathcal{D}^{*\prime}(\Omega'))'_\tau, \mathcal{D}^{*\prime}(\Omega''))$ equipped with the topology of equicontinuous convergence ([16], Proposition 42.2). Since $\mathcal{D}^*(\Omega')$ is reflexive, this is in turn isomorphic to $L_\beta(\mathcal{D}^*(\Omega'), \mathcal{D}^{*\prime}(\Omega''))$. Similarly we have the canonical isomorphism $B^s_\beta(\mathcal{D}^*(\Omega'), \mathcal{D}^*(\Omega'')) = L_\beta(\mathcal{D}^*(\Omega''), \Omega^{*\prime}(\Omega'))$.

Secondly, since $\mathcal{D}^{*\prime}(\Omega')$ and $\mathcal{D}^{*\prime}(\Omega'')$ are complete Grothendieck spaces, we have by Théorème 6 of Grothendieck [3], Chap. II the canonical isomorphism

$$B^s_\varepsilon((\mathcal{D}^{*\prime}(\Omega'))'_{\sigma^*}, (\mathcal{D}^{*\prime}(\Omega'))'_{\sigma^*}) = \mathcal{D}^{*\prime}(\Omega')\,\hat{\otimes}\,\mathcal{D}^{*\prime}(\Omega'').$$

Lastly the multiplication $\mathcal{D}^*(\Omega') \times \mathcal{D}^*(\Omega'') \longrightarrow \mathcal{D}^*(\Omega' \times \Omega'')$, which is separately continuous ([I], Theorem 2.8), induces a linear mapping

$$i : \mathcal{D}^{*\prime}(\Omega' \times \Omega'') \longrightarrow B^s(\mathcal{D}^*(\Omega'), \mathcal{D}^*(\Omega'')).$$

Let K' and K'' be arbitrary compact sets with the cone property included in Ω' and Ω'' respectively. By Théorème 12 of Grothendieck [3] we have the canonical isomorphisms of locally convex spaces:

$$B^s_\beta(\mathcal{D}^*_{K'}, \mathcal{D}^*_{K''}) = B_\beta(\mathcal{D}^*_{K'}, \mathcal{D}^*_{K''}) = (\mathcal{D}^*_{K'} \hat{\otimes} \mathcal{D}^*_{K''})'_\beta .$$

The last space is by Theorem 2.1 isomorphic to $(\mathcal{D}^*_{K' \times K''})'_\beta$ and hence we have

$$B^s_\beta(\mathcal{D}^*_{K'}, \mathcal{D}^*_{K''}) = (\mathcal{D}^*_{K' \times K''})'_\beta . \tag{2.14}$$

Since compact sets of the form $K' \times K''$ form a fundamental system of compact sets in $\Omega' \times \Omega''$, it follows that i is bijective. In fact, suppose that $i(f) = 0$ for an $f \in \mathcal{D}^{*\prime}(\Omega' \times \Omega'')$. Then the restriction of $i(f)$ to $\mathcal{D}^*_{K'} \times \mathcal{D}^*_{K''}$ vanishes and hence by the isomorphism (2.14) the restriction of f to $\mathcal{D}^*_{K' \times K''}$ vanishes. Thus we have $f = 0$. Similarly if $K \in B^s(\mathcal{D}^*(\Omega'), \mathcal{D}^*(\Omega''))$, then its restriction to $\mathcal{D}^*_{K'} \times \mathcal{D}^*_{K''}$ gives rise to an element $f_{K' \times K''}$ of $(\mathcal{D}^*_{K' \times K''})'$. Since $f_{K' \times K''}$ are compatible with restriction, they define an ultradistribution $f \in \mathcal{D}^{*\prime}(\Omega' \times \Omega'')$ such that $K = i(f)$.

Since every bounded set in $\mathcal{D}^*(\Omega')$ etc. is a bounded sets in some $\mathcal{D}^*_{K'}$ etc., the topological isomorphisms (2.14) imply the topological isomorphism

$$B^s_\beta(\mathcal{D}^*(\Omega'), \mathcal{D}^*(\Omega'')) = (\mathcal{D}^*(\Omega' \times \Omega''))'_\beta . \tag{2.15}$$

In case $* = \{M_p\}$, the topological isomorphism

$$(\mathcal{D}^*(\Omega'))' \hat{\otimes} (\mathcal{D}^*(\Omega''))' = (\mathcal{D}^*(\Omega' \times \Omega''))'$$

may also be proved as the dual of (2.5) by Théorème 12 of Grothendieck [3], Chap. II.

3. The structure theorem of ultradistributions with support in a submanifold. Suppose that F is a linear submanifold of $\mathbb{R}^n$. Under a suitable coordinate system it is written

$$F = \{(x, 0);\ x \in \mathbb{R}^{n'},\ 0 \in \mathbb{R}^{n''}\}. \tag{3.1}$$

A point in $\mathbb{R}^n$ is denoted as (x, y) with $x \in \mathbb{R}^{n'}$ and $y \in \mathbb{R}^{n''}$. If Ω is an open set in $\mathbb{R}^n$, we write

$$\Omega' = F \cap \Omega \tag{3.2}$$

and $\mathcal{D}^*(\Omega')$ etc. stand for spaces of functions on Ω' of n' variables. We recall that

$$\mathcal{D}^{*\prime}_F(\Omega) = \{f \in \mathcal{D}^{*\prime}(\Omega);\ \operatorname{supp} f \subset F\}. \tag{3.3}$$

We have the following analogue of the Schwartz structure theorem of distributions with support in a submanifold (Schwartz [11], Théorème 36).

Theorem 3.1. Suppose that M_p satisfies (M.0), (M.1), (M.2) and (M.3) and that F is a linear submanifold as above. Then every $f(x, y) \in \mathcal{D}^{(M_p)\prime}_F(\Omega)$ (resp. $\mathcal{D}^{\{M_p\}\prime}_F(\Omega)$) is uniquely represented as

$$f(x, y) = \sum_\beta f_\beta(x) \otimes D^\beta \delta(y) \tag{3.4}$$

with

$$f_\beta(x) \in \mathcal{D}^{(M_p)\prime}(\Omega') \quad (\text{resp. } \mathcal{D}^{\{M_p\}\prime}(\Omega'))$$

satisfying the following conditions:

For every compact set $K' \subset \Omega'$ with the cone property there are constants L, h and C (resp. and for every $L > 0$ and $h > 0$ there is a constant C) such that

$$\|f_\beta\|_{(\mathcal{D}^{\{M_p\},h}_{K'})'} \leq C L^{|\beta|}/M_{|\beta|}. \tag{3.5}$$

<u>Conversely if a family of ultradistributions</u> $f_\beta \in \mathcal{D}^{*\prime}(\Omega')$ <u>satisfies the above estimates,</u> <u>then</u> (3.4) <u>converges in</u> $\mathcal{D}^{*\prime}(\Omega)$ <u>and represents an</u> $f \in \mathcal{D}^{*\prime}_F(\Omega)$. <u>We have moreover</u>

(3.6) $$\operatorname{supp} f = \bigcup_\beta \operatorname{supp} f_\beta .$$

<u>Proof</u>. We prove the converse part first. Suppose that $\{f_\beta(x)\} \subset \mathcal{D}^{*\prime}(\Omega')$ satisfies the estimates (3.5).

Let K'' be a compact set with the cone property in $\mathbb{R}^{n''}$ such that $K' \times K'' \subset \Omega$ and let $k = (2L)^{-1}$. Then the bilinear functional $f_\beta(x) \otimes D^\beta \delta(y)$ on $\mathcal{D}^*_{K'} \times \mathcal{D}^*_{K''}$ satisfies the estimate

$$\left|\iint \varphi(x)\, \psi(y) f_\beta(x) D^\beta \delta(y)\,dxdy\right| \leq 2^{-|\beta|} C \|\varphi\|_{\mathcal{D}^{\{M_p\},h}_{K'}} \|\psi\|_{\mathcal{D}^{\{M_p\},k}_{K''}} .$$

Thus the right hand side of (3.4) converges absolutely in the norm of $B_\beta(\mathcal{D}^{\{M_p\},h}_{K'}, \mathcal{D}^{\{M_p\},k}_{K''}) = (\mathcal{D}^{\{M_p\},h}_{K'} \hat{\otimes}_\pi \mathcal{D}^{\{M_p\},k}_{K''})'_\beta$.

Hence it follows from Proposition 2.2 that (3.4) converges absolutely in the norm of $(\mathcal{D}^{\{M_p\},\ell}_{K' \times K''})'_\beta$ for an ℓ (resp. for all $\ell > 0$). Since the compact sets of the form $K' \times K''$ form a fundamental system of compact sets in Ω, (3.4) converges in $\mathcal{D}^{*\prime}(\Omega)$. It is known that $\mathcal{D}^{*\prime}_F(\Omega)$ is a closed linear subspace of $\mathcal{D}^{*\prime}(\Omega)$ ([I], Theorem 5.8). Hence the sum belongs to $\mathcal{D}^{*\prime}_F(\Omega)$. We have also the inclusion $\operatorname{supp} f \subset \bigcup \operatorname{supp} f_\beta$.

If (3.4) converges in $\mathcal{D}^{*\prime}(\Omega)$, then we have for every $\varphi(x) \in \mathcal{D}^*(\Omega')$ and β

$$\iint \varphi(x)\, \chi(y) y^\beta f(x,y)\,dxdy = \langle \varphi, f_\beta \rangle \beta! ,$$

where χ is a function in $\mathcal{D}^*(\mathbb{R}^{n''})$ which is equal to one on a neighborhood of 0 and has a sufficiently small support. Hence f_β is uniquely determined by f and has a support included in $\operatorname{supp} f$.

To prove the direct part, let $f \in \mathcal{D}^{*\prime}{}_F(\Omega)$. First we consider the case where $\operatorname{supp} f$ is included in a compact convex set K in Ω'.

Let $\tilde{f}(\zeta, \rho)$, where $\zeta = \xi + i\eta$ and $\rho = \sigma + i\tau$, be the Fourier-Laplace transform of $f(x, y)$. Then it follows from the Paley-Wiener theorem for ultradistributions that there exist constants L and C (resp. for each $L > 0$ there exists a constant C) such that

$$|\tilde{f}(\zeta, \rho)| \leq C \exp\{M(L\zeta) + M(L\rho) + H_K(\zeta)\}.$$

Hence if we write

$$\tilde{f}(\zeta, \rho) = \sum_{\beta} \tilde{f}_\beta(\zeta)\rho^\beta \tag{3.7}$$

with

$$\tilde{f}_\beta(\zeta) = \frac{1}{(2\pi i)^{n''}} \oint \frac{\tilde{f}(\zeta, \rho)}{\rho^\beta \rho_1 \cdots \rho_{n''}} d\rho,$$

then we have

$$\begin{aligned} |\tilde{f}_\beta(\zeta)| &\leq \inf_{r_1, \ldots, r_{n''} > 0} \frac{C \exp M(Lr)}{r^\beta} \exp\{M(L\zeta) + H_K(\zeta)\} \\ &\leq C \exp\{M(L\zeta) + H_K(\zeta)\}(\sqrt{n''}L)^{|\beta|}/M_{|\beta|}. \end{aligned} \tag{3.8}$$

Thus it follows from the converse part of the Paley-Wiener theorem that $\tilde{f}_\beta(\zeta)$ are the Fourier-Laplace transforms of $f_\beta \in \mathcal{D}^{*\prime}{}_K(\Omega')$. Clearly $\tilde{f}_\beta(\zeta)\rho^\beta$ is the Fourier-Laplace transform of $f_\beta(x) \otimes D^\beta \delta(y)$. Estimates (3.8) prove that

$$\sum_\beta \exp\{-M(L\zeta) - M(2\sqrt{n''}L\rho) - H_K(\zeta)\} \tilde{f}_\beta(\zeta)\rho^\beta$$

converges absolutely in the supremum norm. Therefore we have (3.4) by the last part of the Paley-Wiener theorem (cf. [I], Proposition 3.6).

To prove (3.5), let K' be a compact convex set in F. If $\varphi \in \mathcal{D}^{\{M_p\},k}_{K'}$, then we have by Lemma 3.3 of [I]

$$|\tilde{\varphi}(\xi)| \leq |K'| \exp\{-M(\xi/(\sqrt{n'}k))\} \|\varphi\| .$$

Hence we have

$$|\langle\varphi, f_\beta\rangle| \leq C|K'| \|\exp\{-M(\xi/(\sqrt{n'}k)+M(L\xi)\}\|_{L^1(\mathbb{R}^{n'})} \|\varphi\| (\sqrt{n''}L)^{|\beta|}/M_{|\beta|} .$$

By Proposition 3.4 of [I] there is a k (resp. for each $k > 0$ there is an L) such that $\exp\{-M(\xi/(\sqrt{n'}k))+M(L\xi)\}$ is integrable. Consequently we have the estimates (3.5).

When the support of f is arbitrary, we take a partition of unity

$$1 = \sum \chi_j(x)$$

in $\mathcal{D}^*(\Omega')$ such that the convex hull of each $\operatorname{supp} \chi_j$ is included in Ω'. Then each term of

$$f(x, y) = \sum \chi_j(x) f(x, y)$$

has the expansion (3.4). In view of (3.6) we can sum up the coefficients of $D^\beta \delta(y)$ with respect to j and obtain expansion (3.4). Since every compact set K' in Ω' meets only a finite number of $\operatorname{supp} \chi_j$, we have also estimate (3.5).

4. The Whitney extension theorem for ultradifferentiable functions. Let F be a linear submanifold of $\mathbb{R}^n$, let Ω be an open set in $\mathbb{R}^n$ and let $\Omega' = \Omega \cap F$ as in §3. We define

(4.1) $\quad \mathcal{E}^*(\Omega)^F = \{\varphi(x, y) \in \mathcal{E}^*(\Omega);\ D_y^\beta \varphi(x, 0) = 0 \text{ for all } \beta\}$.

Clearly this is a closed linear subspace of $\mathcal{E}^*(\Omega)$.

(4.2) $$\mathcal{D}^*(\Omega)^F = \mathcal{D}^*(\Omega) \cap \mathcal{E}^*(\Omega)^F$$

is also a closed linear subspace of $\mathcal{D}^*(\Omega)$.

We are interested in the quotient space $\mathcal{E}^*(\Omega)/\mathcal{E}^*(\Omega)^F$. To describe it we introduce the space $\mathcal{E}^*_\Omega(\Omega')$ of all arrays $(\varphi_\beta(x); \beta \in \mathbb{N}^{n''})$ of functions $\varphi_\beta(x) \in \mathcal{E}^*(\Omega')$ such that for each compact set K' in Ω' and $h > 0$ there is a constant C (resp. there are constants h and C) satisfying

$$(4.3) \qquad \sup_{x\in K'} |D^\alpha \varphi_\beta(x)| \le C h^{|\alpha+\beta|} M_{|\alpha+\beta|} .$$

We have the following expressions of $\mathcal{E}^*_\Omega(\Omega')$:

$$(4.4) \qquad \mathcal{E}^{(M_p)}_\Omega(\Omega') = \varprojlim_{K'\Subset\Omega'} \varprojlim_{h\to 0} \mathcal{E}^{\{M_p\},h}_\Omega(K') ,$$

$$(4.5) \qquad \mathcal{E}^{\{M_p\}}_\Omega(\Omega') = \varprojlim_{K'\Subset\Omega'} \varinjlim_{h\to\infty} \mathcal{E}^{\{M_p\},h}_\Omega(K') ,$$

where $\mathcal{E}^{\{M_p\},h}_\Omega(K')$ is the Banach space of all infinitely differentiable functions $(D^\alpha\varphi_\beta)$ in the sense of Whitney on the regular compact set K' (see page 41 of [I]) which satisfies (4.3). We introduce in $\mathcal{E}^*_\Omega(\Omega')$ locally convex topologies defined by (4.4) and (4.5).

Similarly we define locally convex spaces $\mathcal{D}^*_\Omega(\Omega')$ by

$$(4.6) \qquad \mathcal{D}^{(M_p)}_\Omega(\Omega') = \varinjlim_{K'\Subset\Omega'} \varprojlim_{h\to 0} \mathcal{D}^{\{M_p\},h}_{\Omega,K'} ,$$

$$(4.7) \qquad \mathcal{D}^{\{M_p\}}_\Omega(\Omega') = \varinjlim_{K'\Subset\Omega'} \varinjlim_{h\to\infty} \mathcal{D}^{\{M_p\},h}_{\Omega,K'} ,$$

where $\mathcal{D}^{\{M_p\},h}_{\Omega,K'}$ is the closed linear subspace of $\mathcal{E}^{\{M_p\},h}(K')$ composed of all $\varphi = (\varphi_\beta)$ such that every component φ_β is extended by zero to a function in $\mathcal{D}^{\{M_p\}}(\mathbb{R}^{n'})$. We note that

$$(4.8) \qquad \operatorname{supp} \varphi = \bigcup_\beta \operatorname{supp} \varphi_\beta$$

is a compact set in Ω' for any $\varphi \in \mathcal{D}^*_\Omega(\Omega')$.

By Proposition 2.4 of [I] the inductive limits relative to h in (4.5) and (4.7) are regular. The inductive limits relative to K' in (4.6) and (4.7) are strict. Hence all spaces are Hausdorff. A bounded set in $\mathcal{D}^{(M_p)}_\Omega(\Omega')$ (resp. $\mathcal{D}^{\{M_p\}}_\Omega(\Omega')$) is a bounded set in some $\mathcal{D}^{(M_p)}_{\Omega,K'}$ (resp. $\mathcal{D}^{\{M_p\},h}_{\Omega,K'}$).

Similarly to Theorem 2.6 of [I] we have the following (cf. [6], Chap. III, § 11).

Proposition 4.1. Suppose that M_p satisfies (M.0), (M.1), (M.2)' and (M.3)'. Then $\mathcal{E}_{\Omega}^{(M_p)}(\Omega')$ is an (FG)-space, $\mathcal{D}_{\Omega}^{(M_p)}(\Omega')$ is an (LFG)-space and $\mathcal{D}_{\Omega}^{\{M_p\}}(\Omega')$ is a (DFG)-space. In particular, these spaces and their strong duals are complete reflexive bornologic Grothendieck spaces.

Next we determine the duals of the above spaces.

Proposition 4.2. Suppose that M_p satisfies (M.0), (M.1), (M.2) and (M.3)'. Then the dual of $\mathcal{D}_{\Omega}^{(M_p)}(\Omega')$ (resp. $\mathcal{D}_{\Omega}^{\{M_p\}}(\Omega')$) is the space of all arrays $(f_\beta(x);\ \beta\in\mathbb{N}^{n''})$ of ultradistributions $f_\beta(x)\in\mathcal{D}^{(M_p)\prime}(\Omega')$ (resp. $\mathcal{D}^{\{M_p\}\prime}(\Omega')$) such that for each compact set K' in Ω' there are constants h, L and C (resp. and each $h>0$ and $L>0$ there is a constant C) satisfying

$$\|f_\beta\|_{(\mathcal{D}^{\{M_p\},h}_{K'})'} \leq C L^{|\beta|}/M_{|\beta|}. \tag{4.9}$$

The dual of $\mathcal{E}^*_{\Omega}(\Omega')$ is the subspace of $(\mathcal{D}^*_{\Omega}(\Omega'))'$ composed of all $f=(f_\beta)$ such that

$$\operatorname{supp} f = \bigcup \operatorname{supp} f_\beta \tag{4.10}$$

is a compact set in Ω'.

The canonical bilinear functional is given by the absolutely convergent series

$$\langle(\varphi_\beta), (f_\beta)\rangle = \sum_\beta \langle\varphi_\beta, f_\beta\rangle . \tag{4.11}$$

Proof. Let $i_\beta : \mathcal{D}^*(\Omega') \to \mathcal{D}^*_{\Omega}(\Omega')$ and $p_\beta : \mathcal{D}^*_{\Omega}(\Omega') \to \mathcal{D}^*(\Omega')$ be the canonical injection and projection. Clearly i_β are continuous. p_β are also continuous because we have by (M.2)

$$h^{|\alpha+\beta|}M_{|\alpha+\beta|} \leq A(hH)^{|\alpha+\beta|}M_{|\alpha|}M_{|\beta|} . \tag{4.12}$$

Let f be a continuous linear functional on $\mathcal{D}^*_{\Omega}(\Omega')$. Then there exist $f_\beta \in \mathcal{D}^{*\prime}(\Omega')$ such that

$$\langle i_\beta(\varphi_\beta), f\rangle = \langle \varphi_\beta, f_\beta\rangle, \quad \varphi_\beta \in \mathcal{D}^*(\Omega') .$$

Since for each $\varphi \in \mathcal{D}^*_{\Omega}(\Omega')$

$$\varphi = \sum_\beta i_\beta \circ p_\beta(\varphi) \tag{4.13}$$

converges absolutely, we have (4.11).

The continuity of f implies that for each compact set K' in Ω' there are constants h and C (resp. and $h > 0$ there is a constant C) such that

$$\begin{aligned}
\left| \sum \langle \varphi_\beta , f_\beta\rangle\right| &\leq C \sup_{\alpha,\beta,x} \frac{|D^\alpha \varphi_\beta(x)|}{h^{|\alpha+\beta|} M_{|\alpha+\beta|}} \\
&\leq C \sup_\beta \frac{1}{h^{|\beta|} M_{|\beta|}} \|\varphi_\beta\|_{\mathcal{D}^{\{M_p\},h}_{K'}}
\end{aligned}$$

for all $(\varphi_\beta) \in \mathcal{D}^*_{\Omega,K'}$. Hence (4.9) follows.

Conversely suppose that (f_β) satisfies (4.9). If $0 < k \leq \min\{h/H, 1/2HL\}$, then we have for $(\varphi_\beta) \in \mathcal{D}^*_{\Omega,K'}$

$$\begin{aligned}
\left|\sum \langle \varphi_\beta, f_\beta\rangle\right| &\leq \sum_\beta \Big(\sup_{\alpha,x} \frac{|D^\alpha \varphi_\beta(x)|}{h^{|\alpha|} M_{|\alpha|}}\Big) CL^{|\beta|}/M_{|\beta|} \\
&\leq \sum_\beta AC(HLk)^{|\beta|} \sup_{\alpha,x} \frac{|D^\alpha \varphi_\beta(x)|}{k^{|\alpha+\beta|} M_{|\alpha+\beta|}} \Big(\frac{Hk}{h}\Big)^{|\alpha|} \\
&\leq 2^{n''} AC \|\varphi\|_{\mathcal{D}^{\{M_p\},k}_{\Omega',K'}} .
\end{aligned}$$

Hence the right hand side of (4.11) is a continuous linear functional on $\mathcal{D}^*_{\Omega}(\Omega')$.

The statement for the dual of $\mathcal{E}^*_{\Omega}(\Omega')$ is proved in the same way as [I], Theorem 5.9.

Similarly to [I], Theorem 5.12 we have

Proposition 4.3. Under the assumption of Proposition 4.2 $\mathcal{E}^{\{M_p\}}_{\Omega}(\Omega')$ and its strong dual are complete reflexive bornologic Grothendieck spaces.

We are now able to prove the Whitney type extension theorem for ultra-differentiable functions.

Let $\iota : \mathcal{E}^*(\Omega)^F \to \mathcal{E}^*(\Omega)$ be the canonical injection and let $\rho : \mathcal{E}^*(\Omega) \to \mathcal{E}^*_{\Omega}(\Omega')$ be the mapping defined by $\rho(\varphi(x, y)) = ((-D_y)^{\beta}\varphi(x, 0))$. Clearly $\rho : \mathcal{E}^*(\Omega) \to \mathcal{E}^*_{\Omega}(\Omega')$ and $\rho : \mathcal{D}^*(\Omega) \to \mathcal{D}^*_{\Omega}(\Omega')$ are continuous linear mappings.

Theorem 4.4. Suppose that M_p satisfies (M.0), (M.1), (M.2) and (M.3). Then

$$0 \longrightarrow \mathcal{E}^*(\Omega)^F \xrightarrow{\iota} \mathcal{E}^*(\Omega) \xrightarrow{\rho} \mathcal{E}^*_{\Omega}(\Omega') \to 0 \tag{4.14}$$

and

$$0 \longrightarrow \mathcal{D}^*(\Omega)^F \xrightarrow{\iota} \mathcal{D}^*(\Omega) \xrightarrow{\rho} \mathcal{D}^*_{\Omega}(\Omega') \to 0 \tag{4.15}$$

are topologically exact sequences of locally convex spaces.

Under the dual ρ' of ρ the strong duals of $\mathcal{E}^*_{\Omega}(\Omega')$ and $\mathcal{D}^*_{\Omega}(\Omega')$ are topologically isomorphic to the linear subspaces $\mathcal{E}^{*\prime}{}_F(\Omega)$ and $\mathcal{D}^{*\prime}{}_F(\Omega)$ of $\mathcal{E}^{*\prime}(\Omega)$ and $\mathcal{D}^{*\prime}(\Omega)$ respectively.

In particular, the set of all functions $\varphi \in \mathcal{E}^*(\Omega)$ (resp. $\mathcal{D}^*(\Omega)$) such that $\operatorname{supp}\varphi \cap F = \emptyset$ is dense in $\mathcal{E}^*(\Omega)^F$ (resp. $\mathcal{D}^*(\Omega)^F$).

Proof. By the definition ι is a topological isomorphism and we have

$$\operatorname{im}\iota = \ker\rho .$$

Next we prove that ρ' is a topological isomorphism onto the closed linear subspace $\mathcal{E}^{*\prime}{}_F(\Omega)$ (resp. $\mathcal{D}^{*\prime}_F(\Omega)$) of $\mathcal{E}^{*\prime}(\Omega)$ (resp. $\mathcal{D}^{*\prime}(\Omega)$).

Clearly $\operatorname{im}\rho'$ is included in the orthogonal space $[\mathcal{E}^*(\Omega)^F]^0$ (resp.

$[\mathcal{D}^*(\Omega)^F]^0$) of $\ker \rho$. It is also clear that

$$[\mathcal{E}^*(\Omega)^F]^0 \subset \mathcal{E}^{*\prime}_F(\Omega) \quad \text{and} \quad [\mathcal{D}^*(\Omega)^F]^0 \subset \mathcal{D}^{*\prime}_F(\Omega).$$

Hence ρ' maps $(\mathcal{E}^*_\Omega(\Omega'))'$ (resp. $(\mathcal{D}^*_\Omega(\Omega'))'$) into $\mathcal{E}^{*\prime}_F(\Omega)$ (resp. $\mathcal{D}^{*\prime}_F(\Omega)$).

If $f = (f_\beta) \in (\mathcal{E}^*_\Omega(\Omega'))'$ (resp. $\mathcal{D}^*_\Omega(\Omega'))'$), then we have for every $\varphi \in \mathcal{E}^*(\Omega)$ (resp. $\mathcal{D}^*(\Omega)$)

$$\begin{aligned}\langle \varphi, \rho'(f)\rangle &= \langle \rho(\varphi), f\rangle \\ &= \sum_\beta \langle (-D_y)^\beta \varphi(x, 0), f_\beta(x)\rangle \\ &= \sum_\beta \langle \varphi(x, y), f_\beta(x) \otimes D^\beta \delta(y)\rangle .\end{aligned}$$

This shows that

$$\rho'((f_\beta)) = \sum_\beta f_\beta(x) \otimes D^\beta \delta(y). \tag{4.16}$$

In particular, ρ' is injective. Theorem 3.1 together with Proposition 4.2 shows that $\operatorname{im} \rho'$ coincides with $\mathcal{E}^{*\prime}_F(\Omega)$ (resp. $\mathcal{D}^{*\prime}_F(\Omega)$). Its proof shows also that $(\rho')^{-1}$ is continuous. In fact, if a compact convex set K' in Ω' is fixed, the topologies on $\mathcal{D}^{*\prime}_{K'}$ induced from $\mathcal{E}^{*\prime}(\Omega)$ and $\mathcal{D}^{*\prime}(\Omega)$ coincide and make $\mathcal{D}^{(M_p)\prime}_{K'}$ a (DFG)-space and $\mathcal{D}^{\{M_p\}\prime}_{K'}$ an (FG)-space. In view of the last part of Theorem 1.1 we see from the proof of Theorem 3.1 that $(\rho')^{-1}$ which assigns to $f = \sum f_\beta \otimes D^\beta \delta$ the components (f_β) is continuous. Since the multiplication by a partition of unity is obviously continuous, $(\rho')^{-1}$ is continuous in all cases.

Since $\mathcal{E}^*(\Omega)$, $\mathcal{E}^*_\Omega(\Omega')$, $\mathcal{D}^*(\Omega)$ and $\mathcal{D}^*_\Omega(\Omega')$ are reflexive spaces, the mapping ρ may be regarded as the bidual $(\rho')'$. Hence it follows from the Hahn-Banach theorem that ρ is surjective. Since $\mathcal{E}^*(\Omega)$ etc. have the Mackey topologies, ρ is also a homomorphism ([6], Theorem 3.7).

In the course of proof we have shown that $[\mathcal{E}^*(\Omega)^F]^0 = \mathcal{E}^{*\prime}_F(\Omega)$ (resp.

$[\mathcal{D}^*(\Omega)^F]^0 = \mathcal{D}^{*\prime}_F(\Omega))$. Since $\mathcal{E}^{*\prime}_F(\Omega)$ (resp. $\mathcal{D}^{*\prime}_F(\Omega)$) is the orthogonal space of the space D of all $\varphi \in \mathcal{E}^*(\Omega)$ (resp. $\mathcal{D}^*(\Omega)$) with $\operatorname{supp} \varphi \cap F = \emptyset$, this proves that D is dense in $\mathcal{E}^*(\Omega)^F$ (resp. $\mathcal{D}^*(\Omega)^F$) by the bipolar theorem.

Let A_p be a sequence of positive numbers such that A_p and $A_p/p!$ are logarithmically convex and that

$$\varliminf_{p\to\infty} \left(\frac{M_p}{A_p}\right)^{1/p} > 0 . \tag{4.17}$$

Then Roumieu [10] proves that the space of ultradifferentiable functions of class $\{M_p\}$ is invariant under ultradifferentiable coordinate transformations Φ of class $\{A_p\}$. Similarly we can prove that the ultradifferentiable functions of class (M_p) are stable under coordinate transformations of class (A_p). If

$$\varliminf_{p\to\infty} \left(\frac{M_p}{A_p}\right)^{1/p} = \infty , \tag{4.18}$$

we can also prove that the ultradifferentiable functions of class (M_p) are stable under coordinate transformations of class $\{A_p\}$. Since $A_p = p!$ satisfies (4.18) for all M_p, we see in particular that the spaces $\mathcal{E}^{(M_p)}$, $\mathcal{E}^{\{M_p\}}$, $\mathcal{D}^{(M_p)}$ and $\mathcal{D}^{\{M_p\}}$ are always invariant under real analytic coordinate transformations.

The isomorphisms on spaces $\mathcal{E}^*(\Omega)$ and $\mathcal{D}^*(\Omega)$ onto $\mathcal{E}^*(\Phi^{-1}(\Omega))$ and $\mathcal{D}^*(\Phi^{-1}(\Omega))$ are shown to be topological isomorphisms. Hence we have also isomorphisms of spaces of ultradistributions.

We will say that a submanifold F of $\mathbb{R}^n$ with boundary is <u>sufficiently smooth</u> if there is a sequence A_p satisfying the above conditions and at each point $x \in F$ there is a local coordinate system $\varphi_j(x)$ of class $\{A_p\}$ or (A_p) which maps F onto a neighborhood of zero in a linear submanifold or a half linear submanifold.

Theorem 4.5. Suppose that M_p satisfies (M.0), (M.1), (M.2) and (M.3) and that F is a sufficiently smooth submanifold with boundary of an open set Ω in $\mathbb{R}^n$. If $\varphi = (D^\alpha \varphi)$ is an infinitely differentiable function in the sense of Whitney defined on F and if for each regular compact set K in F and $h > 0$ there is a constant C (resp. there are constants h and C) such that

(4.19) $$\sup_{x \in K} |D^\alpha \varphi(x)| \leq C h^{|\alpha|} M_{|\alpha|}, \quad |\alpha| = 0, 1, 2, \dots ,$$

then there is an ultradifferentiable function $\psi \in \mathcal{E}^{(M_p)}(\Omega)$ (resp. $\mathcal{E}^{\{M_p\}}(\Omega)$) such that

(4.20) $$D^\alpha \varphi = D^\alpha \psi|_F .$$

Proof. First we consider the case where F has no boundary. Then at each point $x \in F$ we can find a sufficiently smooth coordinate system which maps a neighborhood of x in F onto a linear submanifold. Applying Theorem 4.4, we can find an ultradifferentiable function ψ_x of class (M_p) (resp. $\{M_p\}$) defined on a neighborhood U_x of x in Ω which extends $\varphi|_{F \cap U_x}$.

We take a partition of unity

$$1 = \sum \chi_j(x)$$

on Ω subordinate to the covering $\{U_x\} \cup \{\Omega \setminus F\}$. Then

$$\psi(x) = \sum \chi_j(x) \psi_{x_j}(x)$$

gives the desired function.

When F has the boundary ∂F, we construct an ultradifferentiable function ψ_1 on Ω such that

$$D^\alpha \psi_1|_{\partial F} = D^\alpha \varphi|_{\partial F} .$$

Then the function $\varphi_1 = (D^\alpha \varphi - D^\alpha \psi_1)$ on F vanishes on ∂F together with all the derivatives. Hence it can be continued by zero beyond the boundary ∂F. Then we can apply the first method and obtain an extension ψ_2 on Ω. $\psi = \psi_1 + \psi_2$ gives the desired extension.

For each open set Ω in $\mathbb{R}^n$ we can find an increasing sequence K_n of compact sets with real analytic boundary such that $\Omega = \bigcup_n \operatorname{int} K_n$. Hence we have by Yoshinaga's criterion of (DLFG)-spaces ([6], Theorem 11.6) the following.

<u>Theorem 4.6</u>. <u>Suppose that</u> M_p <u>satisfies</u> (M.0), (M.1), (M.2) <u>and</u> (M.3). <u>Then</u> $\mathcal{E}^{\{M_p\}}(\Omega)$ <u>is a (DLFG)-space and hence</u> $\mathcal{E}^{\{M_p\}'}(\Omega)$ <u>is an (LFG)-space.</u>

References

[1] L. Carleson, On universal moment problems, Math. Scand., 9 (1961), 197-206.

[2] A. Grothendieck, Sur les espaces (F) et (DF), Summa Brasil. Math., 3 (1954), 57-122.

[3] A. Grothendieck, Produits Tensoriels Topologiques et Espaces Nucléaires, Mem. Amer. Math. Soc., No.16, Amer. Math. Soc., Providence, 1955.

[4] L. Hörmander, An Introduction to Complex Analysis in Several Variables, Van Nostrand, Princeton, 1966.

[5] = [I] H. Komatsu, Ultradistributions, I, Structure theorems and a characterization, J. Fac. Sci. Univ. Tokyo, Sec. IA, 20 (1973), 25-105.

[6] H. Komatsu, Theory of Locally Convex Spaces, Dept. of Math., Univ. of Tokyo, 1974.

[7] M. Neymark, On the Laplace transform of functionals on classes of infinitely differentiable functions, Ark. f. Mat., 7 (1968), 577-594.

[8] A. Pietsch, Nukleare Lokalkonvexe Räume, Akademie-Verlag, Berlin, 1969.

[9] C. Roumieu, Sur quelques extensions de la notion de distribution, Ann. Ecole Norm. Sup. 3 77 (1960), 41-121.

[10] C. Roumieu, Ultra-distributions définies sur R^n et sur certaines classes de variétés différentiables, J. Anal. Math., 10 (1962-63), 153-192.

[11] L. Schwartz, Théorie des Distributions, 3e éd., Hermann, Paris, 1966.

[12] L. Schwartz, Théorie des noyaux, Proc. Intern. Congress Math., Mass., 1950, Vol.1, pp.220-230.

[13] L. Schwartz, Espaces de fonctions différentiables à valeurs vectorielles, J. Anal. Math., 4 (1952), 88-148.

[14] L. Schwartz, Produits Tensoriels Topologiques d'Espaces Vectoriels Topologiques. Espaces Vectoriels Topologiques Nucléaires. Applications,

Séminaire Schwartz (1953-54), Fac. Sci. Paris, 1954.

[15] L. Schwartz, Théorie des distributions à valeurs vectorielles, Ann. Inst. Fourier Grenoble, 7 (1957), 1-141 et 8 (1958), 1-209.

[16] F. Treves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York-London, 1967.